This week

1. Section 9.1: solutions, slope fields, Euler’s method
2. Section 9.2: first-order linear equations
3. Section 9.3: applications
What is a differential equation?

## Definition

- **A (first order) differential equation** is an equation involving an unknown function and its derivatives

  \[ F(x, y, y') = 0 \]

- A **solution** is a function \( y(x) \), that satisfies the differential equation:

  \[ F \left( x, y(x), \frac{dy(x)}{dx} \right) = 0. \]

## Definition

**A normal (first order) differential equation** is an equation of the form

\[ y' = f(x, y) \]
What is a differential equation?

Example:

\[ y' = \cos(x) \]  

\[ (*) \]

- The function \( y(x) = \sin(x) \) is a solution because

\[
\frac{d}{dx} y(x) = \frac{d}{dx} (\sin(x)) = \cos(x).
\]

- Every anti-derivative of \( \cos(x) \) is a solution of \((*)\).

- The solutions of \((*)\) are

\[ y(x) = \sin(x) + C \]

with \( C \) an arbitrary constant.
What is a differential equation?

Example:

\[ y' = 2xy \]

- The function \( y(x) = e^{x^2} \) is a solution because
  \[
  \frac{dy(x)}{dx} = \ldots
  \]

- For every \( C \) the function \( y(x) = Ce^{x^2} \) is a solution:
  \[
  \frac{d}{dx}Ce^{x^2} = \ldots
  \]
What is a differential equation?

\[ F(x, y, y') = 0 \]

- It is easy to check whether given function \( y(x) \) is a solution.
- Solutions of differential equations contain an arbitrary constant \( C \).
- The value of \( C \) can be determined by specifying a value for \( y(0) \).

**Definition**

- An additional condition like \( y(x_0) = y_0 \) where \( x_0 \) and \( y_0 \) are given values is called an initial condition or boundary condition.
- A set of equations of the form

\[
\begin{align*}
F(x, y, y') &= 0, \\
f(x_0) &= y_0,
\end{align*}
\]

is called an initial value problem or boundary value problem.
Slope fields

\[ y(x_0) = y_0 \]

\[ y(x) \text{ is solution of } y' = f(x, y) \]

passing through \( y_0 = y(x_0) \)

The slope of \( \ell \) is \( y'(x_0) = f(x_0, y_0) \)
Slope fields

\[ y' = 0 \]

\[ y(x) = \]
Slope fields

\[ y' = y \]

\[ y(x) = \]
Slope fields

$y' = y - x$

$y(0) = 0 : y(x) =$

$y(0) = 1 : y(x) =$

$y(0) = 2 : y(x) =$

Slopefield of $y' = y - x$
Euler’s method

2.1

\[ y' = f(x, y) \]

- Recall that a derivative is the limit of a difference quotient

\[
\frac{d y}{d x} = \lim_{h \to 0} \frac{y(x + h) - y(x)}{h}
\]

- For small \( h \) we have

\[
\frac{y(x + h) - y(x)}{h} \approx y'(x) = f(x, y(x)),
\]

hence

\[
y(x + h) \approx y(x) + h f(x, y(x)).
\]
Euler’s method

The equation of tangent line $\ell$ is $y = y_0 + (x - x_0)f(x_0, y_0)$.

Approximate $f(x_0 + h)$ with $y_0 + hf(x_0, y_0)$. 
Euler’s method

\[
\begin{cases}
    y' = f(x, y) \\
    y(x_0) = y_0
\end{cases}
\]

- Fix the **step size** \( h \).
- Make a table of points \((x_n, y_n)\), starting with \((x_0, y_0)\), where every point is calculated from the previous one with the equations

\[
\begin{align*}
    x_{n+1} &= x_n + h \\
    y_{n+1} &= y_n + hf(x_n, y_n)
\end{align*}
\]
Euler's method

\[ \frac{dy}{dx} = f(x, y) = y - x \]

\[ y(0) = 0 \]

Choose \( h = 0.5 \).

\[
\begin{array}{c|c|c|c}
 n & x_n & y_n & y(x_n) \\
 \hline
 0 & 0 & 0 & 0 \\
 1 & 0.5 & \_ & \_ \\
 2 & 1.0 & \_ & \_ \\
 3 & 1.5 & \_ & \_ \\
 4 & 2.0 & \_ & \_ \\
 5 & 2.5 & \_ & \_ \\
\end{array}
\]
Euler’s methods

Approximation become better by choosing smaller values for $h$. 

$h = \frac{1}{2}$

$h = \frac{1}{4}$

$h = \frac{1}{8}$
Definition

A **linear first order differential equation** is a differential equation of the form

\[ y' + P(x)y = Q(x) \]

where \( P \) and \( Q \) are functions of \( x \).

- Notice that \( y' = f(x, y) \) with \( f(x, y) = Q(x) - P(x)y \).
- The equation is called **first-order** because it only contains the first derivative of \( y \).
- The equation is called **linear** because there are no nonlinear terms containing \( y \) and \( y' \), such as \( y^2 \) or \( \cos(y') \).
Linear first order differential equations

\[ y' + P y = Q \]

- Assume \( v(x) \) is a function that satisfies the equation
  \[ v' = P v. \] (1)
- Then

\[ \frac{d}{dx} (vy) = \]

- Integrate left- and right-hand side

\[ vy = \]

- Divide left- and right-hand side by \( v \):

\[ y(x) = \]
Equation (1) is a **separable** differential equation that can be solved by integration (see lectures of week 2):

\[ v' = Pv \implies v(x) = e^\int P(x) \, dx, \]

where \( \int P(x) \, dx \) is an anti-derivative of \( P(x) \).

Solving a linear differential equation goes in two steps:

1. **Find the integrating factor** \( v \):
   
   \[ v(x) = e^\int P(x) \, dx. \]

2. **Find the solutions**:
   
   \[ y(x) = \frac{1}{v(x)} \int v(x) Q(x) \, dx. \]
Always check your answer!

\[ y' + P(x)y = Q(x) \] (1)

- For linear first-order differential equations, the solution is always of the form

\[ y(x) = \frac{1}{v(x)} \int v(x)Q(x) \, dx = g(x) + Ch(x). \]

**Check your answer**

- The function \( g(x) \) should satisfy equation (1).

- Function \( h(x) \) should satisfy the equation \( y' + P(x)y = 0 \).

- The differential equation \( y' + P(x)y = 0 \) is called the **complementary equation**.
Example 1

\[ y' - 2x \, y = x \]

- \( P(x) = -2x \) and \( Q(x) = x \).
- \( \int P(x) \, dx = \), hence \( v(x) = \).
- Integrate \( v(x) Q(x) \):
  \[ \int xe^{-x^2} \, dx = \]
- Find \( y \):
  \[ y = \]
Example 2

\[ xy' + 2y = x^3 \quad (x > 0) \]

- Rewrite the equation in the form \( y' + Py = Q \):

\[
\begin{align*}
y' + \frac{2}{x}y &= x^2 \\
\implies P(x) &= \frac{2}{x} \quad \text{and} \quad Q(x) = x^2
\end{align*}
\]

(1)

- Calculate the integrating factor:

\[
\int P(x) \, dx =
\]

\[
v(x) =
\]

- Find \( y \):

\[
y =
\]
Inductor — stores energy in a magnetic field

Resistor — limits the flow of current
Ohm’s law for RL circuits

\[ L \frac{di}{dt} + Ri(t) = V(t) \]

If we apply a constant voltage \( V(t) = V \) and close the circuit at \( t = 0 \) what will happen with the current \( i(t) \)?
\[
\begin{cases}
\frac{di}{dt} + \frac{R}{L}i(t) = \frac{V}{L}, \\
i(0) = 0.
\end{cases}
\]

\[
\int P(t) \, dt = \text{, hence } v(t) =
\]

- Find the general solution:

\[i(t) = \]

- Setting \(i(0) = 0\) we get \(C = \text{, hence}\)

\[i(t) = \]
The differential equation

\[
\begin{cases}
\frac{d i}{d t} + \frac{R}{L} i(t) = \frac{V}{L}, \\
i(0) = 0.
\end{cases}
\]

has the solution

\[
i(t) = \frac{V}{R} \left(1 - e^{-Rt/L}\right)
\]

The current will eventually reach a steady state value

\[
i_s = \lim_{t \to \infty} i(t) = \frac{V}{R}.
\]
Step response: it takes time to reach the steady state current $i_S = V/R$.

- At $t = L/R$ the current is $(1 - 1/e)i_S \approx 0.631i_S$.
- At $t = 3L/R$ about 95% of the steady state current is reached.
- The steady state is reached faster for smaller values of $L/R$. 
Low-pass filters

Subwoofer

Equalizer
Consider a circuit with $R = L = 1$ (to simplify the algebra) and an oscillating voltage source $V(t) = \cos(\omega t)$.

\[
\frac{d}{dt} i(t) + i(t) = \cos(\omega t)
\]

We will show that after a while the solution is

\[
i(t) = \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi)
\]

where $\varphi$ is a phase shift that depends on the frequency $\omega$.

With “after a while” we mean that $i(t) \approx \frac{1}{\sqrt{1 + \omega^2}} \cos(\omega t - \varphi)$ for large values of $t$. 

\[
\frac{d i}{d t} + i(t) = \cos(\omega t)
\]

- \[\int P(t) \, dt = \int 1 \, dt = t, \text{ hence } v(t) = e^t.\]

- Find the general solution:

\[
i(t) = \frac{1}{v(t)} \int v(t) \cos(\omega t) \, dt
\]

\[
= e^{-t} \int e^t \cos(\omega t) \, dt.
\]
- Use integration by parts twice:

\[
\int e^t \cos(\omega t) \, dt = e^t \cos(\omega t) - \int e^t \cdot -\omega \sin(\omega t) \, dt \\
= e^t \cos(\omega t) + \omega \int e^t \sin(\omega t) \, dt \\
= e^t \cos(\omega t) + \omega \left( e^t \sin(\omega t) - \int e^t \cdot \omega \cos(\omega t) \, dt \right) \\
= e^t \cos(\omega t) + \omega e^t \sin(\omega t) - \omega^2 \int e^t \cos(\omega t) \, dt.
\]

- This gives

\[
\int e^t \cos(\omega t) \, dt = e^t \left[ \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) \right] + C.
\]
The general solution is

\[ i(t) = e^{-t} \int e^t \cos(\omega t) \, dt \]

\[ = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t) + Ce^{-t}. \]

For large values of \( t \) the term \( Ce^{-t} \) is small, so we may neglect this term:

\[ i(t) = \frac{1}{1 + \omega^2} \cos(\omega t) + \frac{\omega}{1 + \omega^2} \sin(\omega t). \]